ON CONTINUOUS REPRESENTATIONS OF A SQUARE UPON ITSELF.

By H. L. Smith.

In a recent paper under the above title, H. Tietze* has shown that every one-to-one, continuous representation of a square upon itself which preserves sense is a deformation; that is, it may be regarded as a member of a continuous, one-parameter family of such representations which contains the identity. The greater part of Tietze's paper, which occupies 57 pages, is concerned with the proof of a fundamental lemma, which is done by means of a process of successive approximations. In the present paper that lemma is proved in a simple manner by using the theorem of Schoenflies† that two closed, simply connected regions whose boundaries are simple closed curves can be represented in a one-to-one, continuous fashion upon each other so that the correspondence so determined between their boundaries is a preassigned one-to-one, continuous one.‡

For the sake of brevity and concreteness, use is made in the present paper of metric devices, but it will be readily seen that such devices are not essential. The proof admits also of generalization to n-dimensions, provided that such is the case for the Schoenflies theorem.

§ 1. Preliminary definitions and lemmas.

For the sake of simplicity the theorems of this paper will for the most part be stated for the square

$$\mathfrak{S}: \qquad |x+y| + |x-y| \leqq 2;$$

it will be apparent that they may be generalized to the case of any point set that is in one-to-one, continuous correspondence with the square $\mathfrak{S}$.

A one-to-one, continuous representation R of the square $\mathfrak{S}$ upon itself, which is denoted by the equations

$$R: \qquad \xi = f(x, y), \qquad \eta = g(x, y),$$

is said to be a *deformation* provided there exists a pair of functions $\varphi(x,$

* Rendiconti di Palermo, vol. 38 (1914), p. 247.

† Bericht über die Entwicklung der Lehre von den Punktmannigfaltigkeiten, Part II, p. 109.

‡ Since the present paper was written, the writer has found a paper of Tietze's (Sitzungsberichte der Akademie der Wissenschaften in Wien (1913), p. 1653) in which among others he deduces the Schoenflies theorem. It does not seem to have occurred to him to deduce the Tietze theorem from the theorem of Schoenflies.

$y, t)$, $\psi(x, y, t)$, where, in particular, the range for t may be taken as $0 \leqq t \leqq 1$, and such that the equations

$$\xi = \varphi(x, y, t), \qquad \eta = \psi(x, y, t)$$

define for each t a one-to-one, continuous representation of $\mathfrak{S}$ upon itself, the one corresponding to $t = 0$ being the identity and the one corresponding to $t = 1$ being R, and where, finally, φ and ψ are continuous in (x, y) and t separately. If φ and ψ are continuous in x, y, t simultaneously, then R is a deformation *in the strong sense.*

It is easily shown that the product of two deformations (in the strong sense) of $\mathfrak{S}$ is again a deformation (in the strong sense) of $\mathfrak{S}$.

We also assume the Tietze theorem for the one dimensional cases.*

LEMMA 1. *If $j_\pm$ are two simple curves joining the points $A_\pm$: $(\pm 1, 0)$, having no points in common except their end points $A_\pm$, and lying within $\mathfrak{S}$ with the exception of their end points, then there is a deformation of $\mathfrak{S}$ into itself that leaves the boundary of $\mathfrak{S}$ pointwise invariant and carries j_+ into j_- in such a way as to preserve a preassigned one-to-one, continuous correspondence T between $j_\pm$.*

We assume that the notations for the curves $j_\pm$ are so chosen that the region bounded by the simple closed curve made up of the points of j_+ together with the points of the boundary of $\mathfrak{S}$ whose ordinates are greater than zero contains no point of j_-. Then there exists, by the theorem of Schoenflies above referred to, a one-to-one, continuous representation π of $\mathfrak{S}$ upon itself such that $\pi(j_\pm) = k_\pm$, where $k_\pm$ are defined by the equations†

$$k_\pm: \qquad y = \pm \tfrac{1}{2} \operatorname{sm} (|1 + x|, |1 - x|) = \pm \omega(x),$$

and such that the boundary of $\mathfrak{S}$ is left pointwise invariant. The representation π may (and will) be so chosen that if $P_\pm$ are points of $j_\pm$ which correspond according to T, then $\pi(P_\pm)$ have the same abscissa. In the above sm is an abbreviation for "the smaller."

There exists a deformation in the strong sense of the square $\mathfrak{S}$ which leaves the boundary pointwise invariant and carries k_+ into k_- in such a way that the image of each point has the same abscissa as the point itself. For, if we denote by sgn z, 1, 0, or -1 according as z is positive, zero, or negative, such a deformation is given by the equations‡

* Tietze, loc. cit., p. 249.

† k_+ is clearly the broken line gotten by joining the point pairs $(-1, 0)$, $(0, 1/2)$ and $(0, 1/2)$, $(1, 0)$ by line segments.

‡ This is clearly an up-and-down stretching of the square.

$$D: \quad \begin{cases} \xi = x \\ \eta = \tau[x, y, t, \kappa(y - t\omega(x))] \end{cases} \qquad 0 \leqq t \leqq 1,$$

where

$$\begin{cases} \kappa(p) = \operatorname{sgn} p + 1 - \operatorname{sgn} p^2, \\ \tau(x, y, t, q) = q\left[1 - (1 - yq)\dfrac{1 + tq\omega(x)}{1 - tq\omega(x)}\right]. \end{cases}$$

We can now prove the lemma. For $\pi^{-1}D\pi$ is a deformation which has the properties there described.

LEMMA 2. *There exists a simple curve joining $A_{\pm}$, lying within $\mathfrak{S}$ except for its end points $A_{\pm}$, and having no points in common with $j_{\pm}$ except its end points $A_{\pm}$.**

Let π denote a one-to-one, continuous representation of $\mathfrak{S}$ upon the circle whose equation in polar coördinates is $\rho = 1$, such that $A_{\pm}$ are each invariant. Let J denote the image of the points of $j_{\pm}$. Let $\lambda(t)$ be the least upper bound of values of ρ for points (ρ, t), $0 \leqq t \leqq \pi$, which are projections on the line $\theta = t$ of points of J. Then the equations

$$C: \quad \begin{cases} \rho = \tfrac{1}{2}[1 + \lambda(t)], \\ \theta = t, \end{cases} \qquad 0 \leqq t \leqq \pi$$

define a simple curve† lying in the circle $\rho = 1$ with the exception of its end points $A_{\pm}$, and having no points in common with J except $A_{\pm}$. Then $\pi^{-1}C$ is the curve of the lemma.‡

FUNDAMENTAL LEMMA. *If T is a one-to-one, continuous representation of $\mathfrak{S}$ upon itself which leaves the boundary pointwise invariant, then there exists a deformation D in the strong sense of $\mathfrak{S}$ such that DT^{-1} leaves pointwise invariant the boundary of $\mathfrak{S}$ and the segment l joining $A_{\pm}$.*

Set $T^{-1}(l) = j$. Then by Lemma 2 there is a curve k joining $A_{\pm}$, lying within $\mathfrak{S}$ except for the points $A_{\pm}$, and having no points in common with j, l except $A_{\pm}$. By Lemma 1 if c denotes any correspondence from l to k, there exists a deformation in the strong sense D_1 which carries j into k in such a way as to preserve the correspondence cT on j to k, the boundary of $\mathfrak{S}$ remaining invariant. Further there exists a similar deformation D_2 which carries k into l in such a way as to preserve the correspondence c^{-1} on k to l, the boundary remaining invariant. Then $D = D_2D_1$ is the deformation required in the lemma.

* Professor R. L. Moore has indicated to me how this lemma may be easily proved on the basis of Theorems 44 and 40 of his paper, On the Foundations of Plane Analysis Situs (Transactions of the American Mathematical Society (1916), p. 155). Such a proof is not completely metric.

† This is a consequence of the theorem that if a function $f(xt)$ is uniformly continuous in x and t, then the least upper bound of $f(xt)$ with respect to t is a continuous function of x.

‡ The proof from here on is not essentially different from that of Tietze.

Generalized lemma. *If l and m are two segments which divide a square into four congruent squares, and if T is a one-to-one, continuous representation of that square upon itself which leaves the boundary pointwise invariant, then there exists a deformation Δ in the strong sense of the square such that ΔT^{-1} leaves the boundary of the square, l, and m pointwise invariant.*

By the above fundamental lemma there is a defomation D of the square which is such that DT^{-1} leaves pointwise invariant the boundary and l. Similarly* there is a deformation d_+ of one of the two rectangles into which the square is divided by l, which exists in the strong sense and is such that D_+ (DT^{-1}), leaves pointwise invariant the boundary of the rectangle and the part of m lying within the rectangle. A similar deformation exists for the other rectangle; call it d_-. Combining the ranges of $d_\pm$, we obtain a deformation d over the entire square such that $dD = \Delta$ will serve as the required deformation.

§ 2. Proof of the theorem.

Theorem. *A one-to-one, continuous representation R of a square $\mathfrak{S}$ upon itself which preserves sense on the boundary is a deformation in the strong sense.*

Suppose R leaves the boundary pointwise invariant. Then consider the set of lines

$$L_n: \qquad x = \pm\,.\,a_1 \cdots a_n, \qquad y = \pm\,.\,b_1 \cdots b_n,$$

where the right members of the equations are decimals expressed in the scale of 2, and where L_0 is defined specially to be the pair of lines $x = 0$, $y = 0$.

There exists for each n a deformation Δ_n of $\mathfrak{S}$ in the strong sense which is such that $\Delta_n R^{-1}$ leaves invariant all points of $\mathfrak{S}$ on the boundary and on the lines L_n. This is clearly true for $n = 0$ by the generalized lemma. To prove it for $n + 1$ on the hypothesis that it holds for n, we have only to apply the generalized lemma to each of the squares into which L_n divides $\mathfrak{S}$; the deformations so found when united form a single deformation over the entire square $\mathfrak{S}$, this deformation multiplied on the right by Δ_n gives Δ_{n+1}.

Since the deformation Δ_n is identical with R on the boundary of $\mathfrak{S}$ and on L_n, the limit of Δ_n is R, and a simple construction shows that this limit is a deformation, and further, one in the strong sense.†

The General Case. Suppose T is a one-to-one, continuous representation of $\mathfrak{S}$ upon the circle whose equation in polar coördinates is $\rho = 1$. Set $C = TRT^{-1}$, and if we denote by $\theta' = \nu(\theta)$ the one-to-one,

* In applying the fundamental lemma take the T of that lemma as TD^{-1}.

† Cf. Tietze, loc. cit., p. 253.

continuous representation of the peripherie of the circle upon itself which is defined by C, and if c is the one-to-one, continuous representation of the circle upon itself given by the equations

$$c: \qquad \rho' = \rho, \qquad \theta' = \nu(\theta),$$

then the equation

$$R = (T^{-1}cT)(T^{-1}c^{-1}CT)$$

expresses R as the product of two one-to-one, continuous representations of $\mathfrak{S}$ upon itself, the second of which leaves pointwise invariant the boundary of $\mathfrak{S}$, and is therefore a deformation. That the first one is also a deformation follows readily from the fact that by the Tietze theorem for the one-dimensional case, C, and therefore c, is a deformation. Hence R is a deformation, and the theorem is proved.

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